Lecture 19: Maximum Principles & LaPlace's Eqn.

The LaPlace Eqn

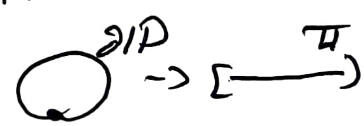
- Any time-independent Solution to the heat-or wave eqn. must have $-\Delta u = 0$. This is the laplace equation. It usually has some B.C. $u|_{\partial U} = f$
- Functions satisfying the LaPlace Eqn. are called harmonic, we will see that they have many nice properties.
- The LaPlace Eqn. Commonly arises in physics. A Conservative v. field may be represented by a gradient $v = \nabla \Phi$. If the vector field is Solenoidal, $\nabla \cdot v = 0$ or $\Delta \Phi = 0$. Similar considerations arise in electrostatics.
- We will focus on $U = \mathbb{D} \subseteq \mathbb{R}^2$. Given $g \in C^0(\partial \mathbb{D})$, we solve

$$\begin{cases} \Delta u = 0 \\ u|_{\partial \mathbb{D}} = g \end{cases}$$

- In the separation of variables in polar coord, we found the harmonic family

$$\Phi_k(r, \theta) = r^{|k|} e^{ik\theta} \quad \text{for } k \in \mathbb{Z}$$

By identifying $\partial \mathbb{D} \leftrightarrow \mathbb{T}$ by



angles, we may write $g = g(\theta)$ as

$$g(\theta) = \sum_{k \in \mathbb{Z}} c_k[g] e^{ik\theta}$$

Since $e^{ik\theta} = \Phi_k(1, \theta)$, we hope to construct a solution

$$u(r, \theta) = \sum_{k \in \mathbb{Z}} c_k[g] \Phi_k(r, \theta)$$

For $g \in C^0$, $\{c_k[g]\}$ is bdd and $|\Phi_k(r, \theta)| = r^{|k|}$ has

$$\sum_{k \in \mathbb{Z}} r^{|k|} < \infty \quad \text{for } r < 1$$

In fact, for $\{r \leq R\}$ and $R < 1$, convergence is uniform.

• Let's try to clean this up.

$$u(r,\theta) = \sum_{k\in\mathbb{Z}} \frac{1}{2\pi}\int_0^{2\pi} r^{|k|} e^{ik(\theta-\eta)} g(\eta)\,d\eta$$

as convergence is uniform in θ for $\{r \le R\}$,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k\in\mathbb{Z}} r^{|k|} e^{ik(\theta-\eta)} g(\eta)\,d\eta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\eta)\, g(\eta)\,d\eta$$

for $\quad P_r(\theta) = \sum_{k\in\mathbb{Z}} r^{|k|} e^{ik\theta}$

the Poisson Kernel

We can deduce directly $\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)\,d\theta = 1$

and $P_r(\theta) = 1 + \sum_{k=1}^{\infty} (re^{i\theta})^k + \sum_{k=1}^{\infty} (re^{-i\theta})^k$

$$= 1 + \frac{re^{i\theta}}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}}$$

$$= \frac{1-r^2}{1-2r\cos(\theta)+r^2}$$

•) As $r \to 1^-$, $P_r(\theta)$ concentrates mass at 0, so we expect $u(r,\theta) \to g(\theta)$ as $r \to 1^-$

$\boxed{\text{Thm } 9.1}$ For $g \in C^0(\partial\mathbb{D})$, $\begin{cases} \Delta u = 0 & \text{in } \mathbb{D} \\ u|_{\partial\mathbb{D}} = g \end{cases}$ admits a classical solution $u \in C^\infty(\mathbb{D}) \cap C^0(\overline{\mathbb{D}})$ given by

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\eta)\, g(\eta)\,d\eta$$

$\boxed{\text{Pf}}$ Since $P_r(\theta)$ is smooth (for $r<1$) and $\Delta P_r(\theta) = 0$, $u(r,\theta)$ is smooth and satisfies $\Delta u(r,\theta) = 0$ as well. We check the boundary condition.

$$\lim_{r\to 1^-} u(r,\theta) = g(\theta)$$

•) We write

$$u(r,\theta) - g(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\eta)[g(\theta-\eta) - g(\theta)]\,d\eta. \quad \text{Pick } \varepsilon>0$$

By continuity, there exists $\delta>0$ s.t. $|g(\theta-\eta)-g(\theta)|<\varepsilon$ for $|\eta|<\delta$.

For $|\eta|\geq\delta$, $\displaystyle\max_{\delta\leq|\eta|\leq\pi} P_r(\eta) = P_r(\delta)$ so

$$|u(r,\theta)-g(\theta)| \leq \frac{1}{2\pi}\left[\int_{-\delta}^{\delta} P_r(\eta)\cdot\varepsilon\,d\eta + \int_{\delta<|\eta|<\pi} P_r(\delta)\,|g(\theta-\eta)-g(\theta)|\,d\eta\right]$$

$$\cancel{\leq \frac{1}{2\pi}\left[\varepsilon(2\pi) + 2\|g\|_\infty \int_{\delta<\eta<\pi} P_r(\delta)\,d\eta\right]}$$

$$\cancel{\leq \varepsilon(\ldots) + 2\|g\|_\infty}$$

$$\leq \frac{1}{2\pi}\left[\varepsilon\cdot 2\pi + 2\|g\|_\infty \int_{\delta<\eta<\pi} P_r(\delta)\,d\eta\right]$$

$$\leq \varepsilon + 2\|g\|_\infty P_r(\delta)$$

Since $\displaystyle\lim_{r\to1^-} P_r(\delta)=0$, for $R<1$ and $R<r<1$,

$$2\|g\|_\infty P_r(\delta) < \varepsilon \quad \text{for } R<r<1.$$

So $|u(r,\theta)-g(\theta)| \leq 2\varepsilon$ for $R<r<1$. $\square$

Hence, $\displaystyle\lim_{r\to1^-} |u(r,\theta)-g(\theta)| = 0.$

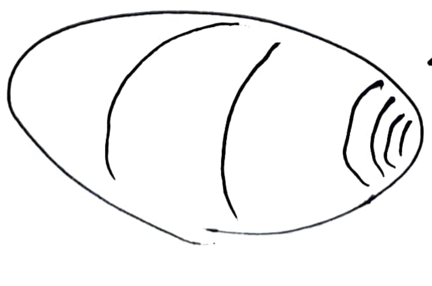
ex.) $g(\theta) = \begin{cases} 1-|\theta|/a & |\theta|\leq a \quad \text{for } a\in(0,\pi) \\ 0 & a<|\theta|<\pi \end{cases}$

"Hot spot on a point of a plate"

$g(\theta)$

gives

$$u(r,\theta) = \frac{a}{2\pi} + \frac{2}{a\pi}\sum_{k=1}^{\infty} \frac{1-\cos(ka)}{k^2}\cos(k\theta)$$



Increasing height of u

Mean Value Formula

•) Setting $r=0$, ~~u(0)~~ $u(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\eta)\,d\eta$
because $P_0(\theta)=1$. This roughly says that the center point is the average of the edge points.

•) Let A_n denote the volume of the unit sphere in $\mathbb{R}^n$!

Notice $\quad \mathrm{Vol}\,[\partial B(x_0; r)] = A_n r^{n-1}$ $\qquad$ ($n-1$ - dim. volume)

$\mathrm{vol}\,[B(x_0; r)] = \frac{A_n}{n} r^n$ $\qquad$ (n-dim. volume)

These will be important to averaging as above. Further, we introduce

$$G_R(x) = \begin{cases} \frac{1}{2\pi}\ln(r/R) & n=2 \\ \frac{1}{(n-2)A_n}\left[\frac{1}{R^{n-2}} - \frac{1}{r^{n-2}}\right] & n\geq 3 \end{cases}$$

the unique solution of

$$\frac{\partial}{\partial r} G_R = \frac{1}{A_n r^{n-1}} \qquad G_R\big|_{r=R} = 0$$

•) Notice also $G_R(x)$ is integrable (radial volume element is $A_n r^{n-1} dr$)

$\boxed{\text{Th}^{\underline{m}}\ 9.3}$ Assume $u \in C^2(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$. For $R>0$ such that $\overline{B(x_0,R)} \subseteq \Omega$

$$u(x_0) = \frac{1}{A_n R^{n-1}}\int_{\partial B(x_0,R)} u(x)\,dS + \int_{B(x_0,R)} G_R(x-x_0)\,\Delta u(x)\,dx$$

$\boxed{\text{Pf}}$ By a change of variables, consider $x_0=0$. Recall Δ has radial component $\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right)$ such that $\Delta G_R(x)=0$. for $x\neq 0$. Then, we consider $\varepsilon<r<R$ for some $\varepsilon>0$ and apply Green's Identity:

$$\int_{\{\varepsilon<r<R\}} G_R\,\Delta u\,dx = \int_{\partial B(0,R)}\left(G_R\,{}^{\partial u}\!/\!_{\partial \nu} - u\,{}^{\partial G_R}\!/\!_{\partial\nu}\right)dS$$
$$- \int_{\partial B(0,\varepsilon)}\left(G_R\,{}^{\partial u}\!/\!_{\partial \nu} - u\,{}^{\partial G_R}\!/\!_{\partial\nu}\right)dS$$

- Since u & G_R are integrable on $B(0;R)$ ~~[illegible]~~, $\partial G_R/\partial r$
~~[illegible]~~,

$$\lim_{\varepsilon \to 0} \int_{\{\varepsilon < r < R\}} G_R \Delta u \, dx = \int_{B(0,R)} G_R \Delta u \, dx$$

Thus, we treat $\int_{\partial B(0,R)} G_R \, \partial u/\partial r - u \, \partial G_R/\partial r \, dS$ (A)
and the integral over $\partial B(0,\varepsilon)$.

First, $(A) = \int_{\partial B(0,R)} 0 \cdot \partial u/\partial r - u \cdot \frac{1}{A_n R^{n-1}} dS = -\frac{1}{A_n R^{n-1}} \int_{\partial B(0,R)} u \, dS$

and Second,

$$\int_{\partial B(0,\varepsilon)} G_R \, \partial u/\partial r - u \, \partial G_R/\partial r \, dS \qquad (B)$$
$$= G_R(\varepsilon) \int_{\partial B(0,\varepsilon)} \partial u/\partial r \, dS + \frac{1}{A_n \varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} u \, dS$$

Notice that $\partial u/\partial r$ & u are bdd on $B(0,R)$, so

$$|(B)| \leq G_R(\varepsilon) A_n \varepsilon^{n-1} + \frac{1}{A_n \varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} u \, dS$$

and $G_R(\varepsilon) A_n \varepsilon^{n-1} \to 0$ as $\varepsilon \to 0$.

By Continuity, $\frac{1}{A_n \varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} u \, dS \to u(0)$ as $\varepsilon \to 0$.

Hence, $\int_{B(0,R)} G_R \Delta u \, dx = u(0) - \frac{1}{A_n R^{n-1}} \int_{\partial B(0,R)} u \, dS$. $\square$

- While the above formula may looks quite odd,
it simplifies to

$$u(x_0) = \frac{1}{A_n R^{n-1}} \int_{\partial B(x,R)} u(x) \, dS$$

when u is harmonic. Thus, it immediately generalizes the circle formula. We may actually squeeze a stronger result out of this.

$\boxed{\text{Corollary 9.4}}$ Suppose $\Omega \subset \mathbb{R}^n$ for $n \geq 2$. For $u \in C^2(\Omega)$,
the following are equivalent. $\hookrightarrow \Omega$ open

(A) $\Delta u = 0$ on Ω

(B) For $\overline{B(x_0,R)} \subset \Omega$, $u(x_0) = \frac{1}{A_n R^{n-1}} \int_{\partial B(x_0;R)} u \, dS$

(C) For $\overline{B(x_0,R)} \subset \Omega$, $u(x_0) = \frac{n}{A_n R^n} \int_{B(x_0,R)} u \, dx$

$\boxed{\text{Pf}}$ A $\Rightarrow$ B as noted above

For B $\Rightarrow$ C, recall that

$$\int_{B(x_0,R)} u \, dx = \int_0^R \int_{\partial B(x_0,r)} u \, dS \, dr$$
$$= \int_0^R (A_n R^{n-1}) u(x_0) dr = \frac{A_n R^n}{n} u(x_0)$$

as desired. The reverse C $\Rightarrow$ B follows similarly by differentiation.

For B $\Rightarrow$ A, the Mean value Formula gives $\int_{B(x_0,R)} G_R(x-x_0) \Delta u(x) dx = 0$ whenever $\overline{B(x_0,R)} \subseteq \Omega$

Without loss of generality, assume that ~~...~~ $\Delta u(x_0) < 0$.
Then, $\exists$ $\varepsilon > 0$ and some ball $B(x_0, \delta)$ so $\Delta u(x) \leq -\varepsilon < 0$ on $B(x_0, \delta)$ by continuity.

Since G_R is strictly negative & decreasing as $r \to 0$,
$\int_{B(x_0,\delta)} G_R(x-x_0) \Delta u(x) dx > -\varepsilon \, G_R|_{r=\delta} > 0$,
a contradiction. We may argue similarly if $\Delta u(x_0) > 0$. $\square$